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THE ALGEBRAIC SOLUTION OF EQUATIONS.

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Let the general equation $f(x) = 0$ represent

$$x^n + A\,x^{n-2} + B\,x^{n-3} + C\,x^{n-4} + \ldots + Tx + U = 0.$$

We shall denote the sum of all the terms of a given kind by Σ; thus,

$$u^2 + v^2 + y^2 + \ldots = \Sigma\,u^2, \qquad uv + uy + uz + \ldots = \Sigma\,uv.$$

We may then write $f(x) = 0$

$$x^n + [a\;\Sigma u^2 + b\;\Sigma\,uv]\,x^{n-2} + [c\;\Sigma u^3 + d\;\Sigma u^2v + e\;\Sigma uvy]\,x^{n-3} + \ldots$$
$$+ \,l\;\Sigma u^n + m\;\Sigma u^{n-1}v + n\;\Sigma u^{n-2}v^2 + \ldots = 0\,,$$

in which the functions of $u, v, y, \ldots, z$ are symmetrical, and each coefficient involves all the possible symmetrical functions of $u, v, y, \ldots, z$ of its order. Further, the order of the coefficients is always such that the terms of the equation shall be homogeneous of the nth degree.

Letting $x = u + v + y + \ldots + z$, we may then determine the values of the constants $a, b, c, \ldots, l, m$ so that $u + v + y + \ldots + z$ shall be a root, and the equation shall reduce identically to zero. Let us examine the case of the cubic equation

$$x^3 + Ax + B = 0\,, \tag{1}$$

in which we have

$$x^3 + [a\;\Sigma u^2 + b\;uv]\,x + c\;\Sigma u^3 + d\;\Sigma u^2v = 0, \tag{2}$$

there being only two variables u and v employed, because there are only two coefficients A and B that are independent. In order that $u + v$ shall be a root, we shall have identically by substituting in (2),

$$a + c + 1 = 0$$
$$a + b + d + 3 = 0.$$

Of the four quantities a, b, c, d in these adjunctive equations, as we shall term them, any two may be made zero, or assigned any values at pleasure; provided only, that in assigning zero values no coefficient be destroyed which would have the effect to impair the generality of the equation. For instance, if a and b were simultaneously made zero, the term containing x would vanish.

$$\begin{array}{lll} \text{We may assume} & a = 0,\ d = 0\,; & \text{whence } c = -1,\ b = -3: \\ & \text{or } b = 0,\ d = 0\,; & \text{"} \quad a = -3,\ c = 2: \\ & \text{or } b = 0,\ c = 0\,; & \text{"} \quad a = -1,\ d = -2\,. \end{array}$$

Writing the general equation (2) with these values of the constants substituted in their order, we have

$$x^3 - 3uv\,x - (u^3 + v^3) = 0\,, \tag{3}$$

$$x^3 - 3(u^2 + v^2)\,x + 2(u^3 + v^3) = 0\,, \tag{4}$$

$$x^3 - (u^2 + v^2)\,x - 2(u^2v + uv^2) = 0\,. \tag{5}$$

Any one of these equations is identically satisfied by substituting $u + v$ for x since $u + v$ is a root. Equating the first of these with (1), evidently,

$$u^3 + v^3 = -B\,, \qquad uv = -\tfrac{1}{3}A\,.$$

Cubing the second of these and eliminating u, we have the sextic

$$v^6 + Bv^3 - \tfrac{1}{27}A^3 = 0\,,$$

from which

$$x = u + v = \sqrt[3]{-\tfrac{1}{2}B + \sqrt{\tfrac{1}{4}B^2 + {}^{1}\,A^3}} + \sqrt[3]{-\tfrac{1}{2}B - \sqrt{\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3}}$$
$$= \theta_1 + \theta_2.$$

These radical functions θ_1, θ_2 will be readily recognized as Cardan's formula, although obtained in a different manner; and, for this reason, we may refer to form (3) as Cardan's form.

Equating (4) with (1), we have

$$u^2 + v^2 = -\tfrac{1}{3}A\,, \qquad u^3 + v^3 = -\tfrac{1}{2}B. \tag{6}$$

Eliminating u, we have the sextic

$$v^6 + \tfrac{1}{2}Av^4 - \tfrac{1}{2}Bv^3 + \tfrac{1}{6}A^2v^2 + \tfrac{1}{2}(\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3) = 0\,, \tag{7}$$

which is neither a cubic nor quadratic in form, but rather a combination of these. Now u and v in these equations do not represent the same radical functions that they do in Cardan's form; namely θ_1 and θ_2, but their sum is still equal to x. So, if u is greater than θ_1, then v is less than θ_2 by the same amount, and we may at once write

$$\begin{aligned} u &= \theta_1 + a\,, \\ v &= \theta_2 - a\,; \end{aligned} \tag{8}$$

and substituting these values in (6), we obtain directly

$$a^2 + (\theta_1 - \theta_2)\, a + \tfrac{1}{2}(\theta_1^2 + \theta_2^2) = \tfrac{1}{2}\, \theta_1 . \theta_2 ,$$

$$3(\theta_1 + \theta_2)\, a^2 + 3(\theta_1^2 - \theta_2^2)\, a + (\theta_1^3 + \theta_2^3) = -\tfrac{1}{2}(\theta_1^3 + \theta_2^3) .$$

Solving either of these quadratics in a, we obtain at once

$$a = -\tfrac{1}{2}(\theta_1 - \theta_2) \pm \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)} .$$

Substituting this value in (8), using plus sign,

$$u = \tfrac{1}{2}(\theta_1 + \theta_2) + \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)} ,$$

$$v = \tfrac{1}{2}(\theta_1 + \theta_2) - \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)} .$$

These values of u and v are algebraic solutions of the curious group (6) which produced upon elimination the irreducible sextic (7).

The six roots of the sextic in Cardan's form are obviously θ_1, $\omega\theta$, $\omega^2\theta_1$, θ_2, $\omega\theta_2$, $\omega^2\theta_2$; and taken in pairs, furnish the three roots of the cubic, as follows:

$$\begin{aligned} x = u + v &= \theta_1 + \theta_2 , \\ \text{“} \quad \text{“} \quad &= \omega\theta_1 + \omega^2\theta_2 , \\ \text{“} \quad \text{“} \quad &= \omega^2\theta_1 + \omega\theta_2 ; \end{aligned}$$

in which ω and ω^2 denote as usual the imaginary roots of $\omega^3 - 1 = 0$. The remaining four roots of the sextic (7) are obviously derived from the above values of u and v by simultaneously substituting $\omega\theta_1$, $\omega^2\theta_2$ and $\omega^2\theta_1$, $\omega\theta_2$ in those radical functions.

If we now equate (5) with (1), we shall have

$$u^2 + v^2 = -A , \qquad u^2v + uv^2 = -\tfrac{1}{2}B ;$$

which will give a different sextic still, whose six roots may be as readily written as in the case of the sextic preceding.

Proceeding to the equation of the fourth degree

$$x^4 + A\,x^2 + B\,x + C = 0 , \tag{9}$$

we shall have the corresponding general equation in x, u, v, and y

$$x^4 + [a\,\Sigma u^2 + b\,\Sigma uv]\,x^2 + [c\,\Sigma u^3 + d\,\Sigma u^2v + e\,uvy]\,x + f\,\Sigma u^4 + g\,\Sigma u^3v + h\,\Sigma u^2v^2 + j\,\Sigma u^2vy = 0 .$$

Substituting for x the value $u + v + y$, we have

$$\begin{aligned} a + c + f + 1 &= 0\,, \\ 2a + b + c + d + g + 4 &= 0\,, \\ 2a + 2b + 2d + h + 6 &= 0\,, \\ 2a + 5b + 2d + e + j + 12 &= 0\,. \end{aligned}$$

Five of the nine constants obtained in this set of adjunctive equations may be made zero, or assigned values at pleasure. The number of groups of nine things, taking five at a time, is 126 ; but a few of these would have to be rejected, in order to preserve the quartic in its general form. We may write a few of these forms, assigning zero always to five of the constants.

Assume a, c, d, g, and j equal zero : evidently

$$f = -1, \quad b = -4, \quad h = 2, \quad e = 8\,;$$

and we have

$$\begin{aligned} x^4 - 4\,(uv + uy + vy)\,x^2 + 8\,uvy\,x - (u^4 + v^4 + y^4) & \\ + 2\,(u^2v^2 + u^2y^2 + v^2y^2) = 0\,. & \qquad (10) \end{aligned}$$

Assume b, c, d, g, j equal zero : evidently

$$a = -2, \quad e = -8, \quad f = 1, \quad h = -2\,;$$

and we have

$$\begin{aligned} x^4 - 2\,(u^2 + v^2 + y^2)\,x^2 - 8\,uvy\,x + (u^4 + v^4 + y^4) & \\ - 2\,(u^2v^2 + u^2y^2 + v^2y^2) = 0\,. & \qquad (11) \end{aligned}$$

If a, c, d, h, j equal zero,

$$f = -1\,, \quad b = -3\,, \quad g = -1\,, \quad e = 3\,;$$

and we have

$$\begin{aligned} x^4 - 3\,(uv + uy + vy)\,x^2 + 3\,uvy\,x - (u^4 + v^4 + y^4) & \\ - 3\,(u^3v + u^3y + uv^3 + uy^3 + v^3y + vy^3) = 0\,. & \end{aligned}$$

We may readily write the remaining forms of the quartic, subject to the restriction mentioned above in the case of the cubic, always assigning zero values to five of the constants. The number of forms would be the same, however, if the constants have no particular values assigned, but be left simply indeterminate. All these forms are theoretically important, as we shall see.

Equating the form (11) with (9), evidently,

$$-2(u^2+v^2+y^2)=A,\quad -8\,uvy=B,\quad u^4+v^4+y^4-2(u^2v^2+u^2y^2+v^2y^2)=C;$$

whence

$$u^2+v^2+y^2=-\tfrac{1}{2}A,$$
$$u^2v^2+u^2y^2+v^2y^2=\tfrac{1}{16}A^2-\tfrac{1}{4}C,$$
$$u^2v^2y^2=\tfrac{1}{64}B^2.$$

From which we have

$$x=u+v+y=\theta_1+\theta_2+\theta_3,$$

where θ_1, θ_2, θ_3 are radical functions of the coefficients expressed by solving the cubic equation,

$$\lambda^6+\tfrac{1}{2}A\,\lambda^4+\tfrac{1}{16}(A^2-4C)\,\lambda^2-\tfrac{1}{64}B^3=0.$$

This cubic is the same as employed by Euler in his reduction of the equation of the fourth degree, and we may refer to (11) as Euler's form.

If, however, we equate (10) and (9) we obtain

$$uv+uy+vy=-\tfrac{1}{4}A,$$
$$u^4+v^4+y^4-2(u^2v^2+u^2y^2+v^2y^2)=-C,$$
$$uvy=\tfrac{1}{8}B.$$

Eliminating u and v from these equations we shall have in general an equation of the 24th degree. It is obvious that the values of u, v, and y are not the same here as in Euler's form, namely, θ_1, θ_2, and θ_3, but their sum is still a root of the quartic.

If u be greater than θ_1, v and y will each be less than θ_2 and θ_3 by increments aggregating the same amount, and we may at once assume

$$u=\theta_1+\alpha+\beta,\quad v=\theta_2-\alpha,\quad y=\theta_3-\beta.$$

Substituting these values, as in the case of the cubic in the forms in question, and solving for α and β, we may proceed to write all the roots of the form, or the resolvent, in simple radical functions of the coefficients. Evidently the number of possible forms for the equation of the fourth degree is very large, possibly near a hundred. The elimination of u and v in one case, namely,

$$x^4-6(u^2+v^2+y^2)x^2+8(u^3+v^3+y^3)x-3(u^4+v^4+y^4)$$
$$+6(u^2v^2+u^2y^2+v^2y^2)=0,$$

gives a resolvent of the 12th degree. There may be others. As u, v, and y

change for each form the values of α and β have to be determined in each case in order to ascertain the roots of that particular form.

With only one other application of the adjunctive equations, which is introduced here for purposes of analogy hereafter, I take leave of the theory of the biquadratic.

If we consider the form

$$x^4 + A\,x + B = 0\,,$$

the adjunctive equations become, putting a, b, e, j, and the variable y, equal zero,

$$c + f + 1 = 0\,,$$
$$c + d + g + 4 = 0\,,$$
$$2\,d + h + 6 = 0\,.$$

Assuming the two constants d and g equal zero, we have at once

$$c = -4\,,\quad f = 3\,,\quad h = -6\,,$$

and the equation in u and v becomes

$$x^4 - 4\,(u^3 + v^3)\,x + 3\,(u^4 + v^4) - 6\,u\,v^2 = 0\,.$$

Whence, equating,

$$-4\,(u^3 + v^3) = A\,,\quad 3\,(u^4 + v^4) - 6\,u^2v^2 = B\,;$$

or, simplifying,

$$u^2 - v^2 = P,$$
$$u^3 + v^3 = Q\,,$$

which presents an important analogy with the corresponding form

$$u^2 + v^2 = P,$$
$$u^3 + v^3 = Q,$$

already discussed in the case of the cubic equation. It will be observed that these forms involve the general solution of the fourth and third degrees, the first form solving the quartic, the second the cubic. The determination of u and v in radicals has already been given.

Proceeding to the study of the general equation of the fifth degree,

$$x^5 + A\,x^3 + B\,x^2 + C\,x + D = 0\,,$$

the corresponding general equation becomes

$$\begin{aligned} x^5 &+ [a\,\Sigma u^2 + b\,\Sigma uv]\,x^3 + [c\,\Sigma u^3 + d\,\Sigma u^2v + e\,\Sigma uvy]\,x^2 \\ &+ [f\,\Sigma u^4 + g\,\Sigma u^3v + h\,\Sigma u^2v^2 + j\,\Sigma u^2vy + k\,uvyz]\,x \\ &+ l\,\Sigma u^5 + m\,\Sigma u^4v + n\,\Sigma u^3v^2 + p\,\Sigma u^3vy + q\,\Sigma u^2v^2y + r\,\Sigma u^2vyz = 0\,. \end{aligned}$$

Assuming

$$x = u + v + y + z,$$

we obtain directly

$$\begin{aligned} a + c + f + l + 1 &= 0\,, \\ 3a + b + 2c + d + f + g + m + 5 &= 0\,, \\ 4a + 3b + c + 3d + g + h + n + 10 &= 0\,, \\ 6a + 7b + 2c + 4d + e + g + j + p + 20 &= 0\,, \\ 6a + 12b + 8d + 2e + h + 2j + q + 30 &= 0\,, \\ 6a + 27b + 7e + 3j + k + r + 60 &= 0\,. \end{aligned}$$

There being four independent coefficients, we assume x equal to as many independent variables; but, if the number of coefficients be diminished, we diminish the number of variables correspondingly. If, for example, the coefficient A be assumed equal to zero, then will z be zero; and if B be also zero, we may put y equal zero. If one coefficient and one variable be put equal to zero, obviously only the first five of the above six adjunctive equations will occur; if two coefficients and two variables be each put equal zero, only the first three of the adjunctive equations will appear, and the quintic will be identically satisfied.

We may now deduce a remarkably simple form for the equation of the fifth degree as viewed from several aspects. If A and B be each assumed equal to zero, the general equation becomes the Jerrardian

$$x^5 + Cx + D = 0\,.$$

We may assume y and z equal to zero, the constants a, b, c, d, e, j, k, p, q, r become zero, and the general equation in x, u, and v reduces to

$$x^5 + [f\,\Sigma u^4 + g\,\Sigma u^3v + h\,u^2v^2]\,x + l\,\Sigma u^5 + m\,\Sigma u^4v + n\,\Sigma u^3v^2 = 0\,.$$

The remaining adjunctive equations also become

$$\begin{aligned} f + l + 1 &= 0\,, \\ f + g + m + 5 &= 0\,, \\ g + h + n + 10 &= 0\,. \end{aligned}$$

Of these six constants f, g, h, l, m, n we may assume m, n, and g equal to zero; whence

$$f = -5, \quad h = -10, \quad l = 4,$$

and the above equation becomes

$$x^5 + [-5(u^4 + v^4) - 10u^2v^2]x + 4(u^5 + v^5) = 0.$$

Equating with the above Jerrardian,

$$-5(u^4 + v^4) - 10u^2v^2 = C, \quad 4(u^5 + v^5) = D;$$

whence, reducing,

$$u^2 + v^2 = P,$$
$$u^5 + v^5 = Q;$$

and the solution of this form for u and v, as in its analogous form for the cubic (6), involves the reduction of the general equation of the fifth degree. This, apparently, is a simplification of the Jerrardian form; for, if we substitute $u + v$ for x in any Jerrardian, it asserts only one relation between the variables, and it is visibly non-decomposable into binomial forms.

Common Resolvent for the Fifth and Sixth Degrees.

In the quintic adjunctive group assume $a, b, d, e, f, g, h, j, k, n, p, q, r$, and y and z equal to zero, whence

$$c = -10, \quad l = 9, \quad m = 15,$$

and the Jerrardian

$$x^5 + Bx^2 + D = 0$$

becomes

$$x^5 - 10(u^3 + v^3)x^2 + 9(u^5 + v^5) + 15uv(u^3 + v^3) = 0.$$

Whence, by simplifying,

$$\left.\begin{aligned} u^3 + v^3 &= P, \\ u^5 + v^5 + \tfrac{5}{3}uv(u^3 + v^3) &= Q. \end{aligned}\right\} \qquad (a)$$

The diminutive group for the general sextic

$$x^6 + Ax + B = 0,$$

for the form

$$x^6 + [a\Sigma u^5 + b\Sigma u^4v + c\Sigma u^3v^2]x + d\Sigma u^6 + e\Sigma u^5v + f\Sigma u^4v^2 + gu^3v^3 = 0,$$

is

$$a + d + 1 = 0,$$
$$a + b + e + 6 = 0,$$
$$b + c + f + 15 = 0,$$
$$2c + g + 20 = 0.$$

Assuming c, e, f equal to zero, we have

$$a = 9\,,\quad b = -15\,,\quad d = -10\,,\quad g = -20\,;$$

and substituting and equating, as usual

$$9\,(u^5 + v^5) - 15\,uv\,(u^3 + v^3) = A\,;\quad -10\,(u^6 + v^6) - 20\,u^3v^3 = B\,.$$

Whence

$$\left.\begin{aligned} u^3 + v^3 &= P\,, \\ u^5 + v^5 - \tfrac{5}{3}\,uv\,(u^3 + v^3) &= Q\,. \end{aligned}\right\} \qquad (\beta)$$

These equations (α) and (β) are identical, excepting the change of sign, (α) involving the general solution of the fifth degree, (β) that of the sixth degree. This analogy of the change of sign was pointed out for the third and fourth degrees, where the resolvent for both forms was of the sixth degree, upon which resolvent the solution of both equations depended.

If from (α) we eliminate u, the resolvent is of the twenty-fourth degree, which is directly convertible into the resolvent for (β) by simply changing the sign of the coefficient; from which we conclude, that the solution of the fifth and sixth degrees depends upon a common resolvent equation of the twenty-fourth degree.